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Some Differential Equations Connected with Hypersurfaces.

BY G. O. JAMES.

§1. Surfaces in ordinary space may be treated from two essentially different standpoints. Either we may suppose a surface given by an equation between the cartesian coordinates of a point on it and deduce its properties by the methods of analytic geometry, or we may suppose the infinite system of which it is a member given by a binary differential quadratic form, the linear element, and deduce its properties by the methods of differential geometry. Similarly, curved spaces of any number of dimensions immersed in homaloidal space may be studied from either standpoint. To any algebraic equation connecting the cartesian coordinates of a point in three dimensional Euclidian space, there corresponds a surface, but in order that there may exist a system of surfaces admitting a binary differential quadratic form as linear element, the coefficients of this form must satisfy certain algebraic and differential relations, namely, the discriminant must be positive, i. e. the form itself must be *definite*, and the equations of Gauss and Codazzi must be satisfied. In the same way, in Euclidian space of any number of dimensions, a manifold of points may be separated from the space by an algebraic equation satisfied by the coordinates of a point in this space, but given an n -ary differential quadratic form, its coefficients must satisfy certain relations in order that there may exist a corresponding manifold admitting this as linear element.

These relations have been investigated from a purely algebraic standpoint by Ricci,* who solves the following problem :

* Ricci, "Principii di una teoria delle forme differenziale quadratiche," Annale di Mat., Serie II, Vol. 12.

To determine the conditions that the n -ary differential quadratic form

$$f = \sum_{r,s}^n a_{rs} dx_r dx_s$$

shall be reducible, i. e. can be put identically in the form

$$\sum_{i=1}^{n-1} b_{im} du_i du_m,$$

where u_i is a function of $x_1 \dots x_n$ and b_{im} is a function of $u_1 \dots u_{n-1}$. When f is irreducible, Schlaefli* has shown that it can be deduced from

$$ds^2 = \sum_{i=1}^{n+h} dy_i^2,$$

where

$$0 \leq h \leq \frac{n(n-1)}{2}.$$

Ricci defines b as the class of f and investigates the conditions under which f shall be of a given class. Now, since the linear element of the curved space, or manifoldness of n dimensions, is irreducible and of the first class, the problem comes algebraically to the discussion of the conditions under which f shall be of the first class, and Ricci has completely solved this. The object of this paper is to treat the problem as a problem of differential geometry and not as a part of the theory of quadratic forms. For this I shall suppose the curved space given by an algebraic equation between the cartesian coordinates of a point on it and shall deduce the conditions which the coefficients of the first and second fundamental forms must satisfy, and the differential equations on the integration of which the effective determination of the hypersurface depends. From these is derived the theorem of Beez, that a curved space of dimensions greater than two cannot be deformed so as to preserve its linear element, and hence is only capable of translation and rotation in hyperspace if *inextensible*.

To render the analytic work more manageable, I shall confine myself to four dimensions, and shall adopt the nomenclature of Poincaré in his memoir,† “Sur les Residus des Integrales Doubles.” Four dimensional homaloidal or Euclidian space will then be termed *hyperspace*, and a single relation between the coordinates of a point in hyperspace will define a *hypersurface*.

* *Annale di Mat.*, Serie II, Vol. 5, p. 190.

† *Acta Math.*, t. 9, p. 325.

§2. Suppose the hypersurface to be given in the form

$$F(y_1 y_2 y_3 y_4) = 0$$

of linear element

$$f = ds^2 = \Sigma dy^2.$$

Expressing the orthogonal coordinates in terms of three independent parameters, x_1, x_2, x_3 , by the functions

$$y_i = y_i(x_1 x_2 x_3), \quad i = 1, 2, 3, 4,$$

which, together with their first, second and third partial derivatives, are supposed uniform finite and continuous throughout the region of variation of x_1, x_2 and x_3 , the linear element takes the form

$$f = ds^2 = \sum_{r=1}^3 E_{rs} dx_r dx_s,$$

where

$$E_{rs} = \sum_{i=1}^4 \frac{\partial y_i}{\partial x_r} \frac{\partial y_i}{\partial x_s}.$$

After the analogy of the theory of surfaces, I shall call this ternary differential quadratic form the *first fundamental form* of the hypersurface.

§3. For a *real* hypersurface, the discriminant of f cannot vanish in general,

$$\Delta = |E_{rs}| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \frac{\partial y_3}{\partial x_1} & \frac{\partial y_4}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_4}{\partial x_2} \\ \frac{\partial y_1}{\partial x_3} & \frac{\partial y_2}{\partial x_3} & \frac{\partial y_3}{\partial x_3} & \frac{\partial y_4}{\partial x_3} \end{vmatrix}^2 = \Sigma \begin{vmatrix} \frac{\partial y_i}{\partial x_1} & \frac{\partial y_j}{\partial x_1} & \frac{\partial y_k}{\partial x_1} \\ \frac{\partial y_i}{\partial x_2} & \frac{\partial y_j}{\partial x_2} & \frac{\partial y_k}{\partial x_2} \\ \frac{\partial y_i}{\partial x_3} & \frac{\partial y_j}{\partial x_3} & \frac{\partial y_k}{\partial x_3} \end{vmatrix}^2$$

($i, j, k = 1, 2, 3, 4,$ $i \neq j \neq k$),

and if $\Delta = 0$, the terms of the second member must separately vanish, which necessitates relations among the coordinates other than the equation of the hypersurfaces. Δ is therefore, in general, positive and different from zero, and f is *definite*.

§4. Applying the formula defining the Christoffel symbols of the first kind,

$$\left[\begin{matrix} ik \\ l \end{matrix} \right] = \frac{1}{2} \left(\frac{\partial E_{il}}{\partial x_k} + \frac{\partial E_{kl}}{\partial x_i} - \frac{\partial E_{ik}}{\partial x_l} \right),$$

we have the following eighteen symbols of the first kind of three indices for the case of three parameters :

$$\begin{aligned}
\begin{bmatrix} 11 \\ 1 \end{bmatrix} &= \frac{1}{2} \frac{\partial E_{11}}{\partial x_1}; & \begin{bmatrix} 12 \\ 1 \end{bmatrix} &= \frac{1}{2} \frac{\partial E_{11}}{\partial x_2}; & \begin{bmatrix} 13 \\ 1 \end{bmatrix} &= \frac{1}{2} \frac{\partial E_{11}}{\partial x_3}; & \begin{bmatrix} 22 \\ 1 \end{bmatrix} &= \frac{\partial E_{12}}{\partial x_2} - \frac{1}{2} \frac{\partial E_{22}}{\partial x_1}; \\
\begin{bmatrix} 23 \\ 1 \end{bmatrix} &= \frac{1}{2} \left(\frac{\partial E_{12}}{\partial x_3} + \frac{\partial E_{13}}{\partial x_2} - \frac{\partial E_{23}}{\partial x_1} \right); & \begin{bmatrix} 33 \\ 1 \end{bmatrix} &= \frac{\partial E_{13}}{\partial x_3} - \frac{1}{2} \frac{\partial E_{33}}{\partial x_3}; \\
\begin{bmatrix} 11 \\ 2 \end{bmatrix} &= \frac{\partial E_{12}}{\partial x_1} - \frac{1}{2} \frac{\partial E_{11}}{\partial x_2}; & \begin{bmatrix} 12 \\ 2 \end{bmatrix} &= \frac{1}{2} \frac{\partial E_{22}}{\partial x_1}; & \begin{bmatrix} 13 \\ 2 \end{bmatrix} &= \frac{1}{2} \left(\frac{\partial E_{12}}{\partial x_3} + \frac{\partial E_{23}}{\partial x_1} - \frac{\partial E_{13}}{\partial x_2} \right); \\
\begin{bmatrix} 22 \\ 2 \end{bmatrix} &= \frac{1}{2} \frac{\partial E_{22}}{\partial x_2}; & \begin{bmatrix} 23 \\ 2 \end{bmatrix} &= \frac{1}{2} \frac{\partial E_{22}}{\partial x_3}; & \begin{bmatrix} 33 \\ 2 \end{bmatrix} &= \frac{\partial E_{23}}{\partial x_3} - \frac{1}{2} \frac{\partial E_{33}}{\partial x_2}; \\
\begin{bmatrix} 11 \\ 3 \end{bmatrix} &= \frac{\partial E_{13}}{\partial x_1} - \frac{1}{2} \frac{\partial E_{11}}{\partial x_3}; & \begin{bmatrix} 12 \\ 3 \end{bmatrix} &= \frac{1}{2} \left(\frac{\partial E_{13}}{\partial x_2} + \frac{\partial E_{23}}{\partial x_1} - \frac{\partial E_{12}}{\partial x_3} \right); & \begin{bmatrix} 13 \\ 3 \end{bmatrix} &= \frac{1}{2} \frac{\partial E_{33}}{\partial x_1}; \\
\begin{bmatrix} 22 \\ 3 \end{bmatrix} &= \frac{\partial E_{23}}{\partial x_2} - \frac{1}{2} \frac{\partial E_{22}}{\partial x_3}; & \begin{bmatrix} 23 \\ 3 \end{bmatrix} &= \frac{1}{2} \frac{\partial E_{33}}{\partial x_2}; & \begin{bmatrix} 33 \\ 3 \end{bmatrix} &= \frac{1}{2} \frac{\partial E_{33}}{\partial x_3}.
\end{aligned}$$

The symbols of the second kind of three indices for a triply orthogonal system of parametric lines will be useful. Making

$$E_{12} = E_{13} = E_{23} = 0$$

in the symbols of the first kind, these are given at once, and to get those of the second kind it is sufficient to observe that

$$A_{\nu\nu} = \frac{1}{E_{\nu\nu}}, \quad A_{\nu\lambda} = 0, \quad \nu \neq \lambda,$$

where A_{ij} is the algebraic complement of a_{ij} divided by a , and $E_{rs} = a_{rs}$. Applying the formulæ

$$\left\{ \begin{matrix} ix \\ \nu \end{matrix} \right\} = \sum_l A_{\nu l} \left[\begin{matrix} ix \\ l \end{matrix} \right],$$

we have

$$\begin{aligned}
\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} &= \frac{1}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{11}}}{\partial x_1}; & \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} &= \frac{1}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{11}}}{\partial x_2}; & \left\{ \begin{matrix} 13 \\ 1 \end{matrix} \right\} &= \frac{1}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{11}}}{\partial x_3}; \\
\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} &= \frac{\sqrt{E_{22}}}{E_{11}} \frac{\partial \sqrt{E_{22}}}{\partial x_1}; & \left\{ \begin{matrix} 23 \\ 1 \end{matrix} \right\} &= 0; & \left\{ \begin{matrix} 33 \\ 1 \end{matrix} \right\} &= -\frac{\sqrt{E_{33}}}{E_{11}} \frac{\partial \sqrt{E_{33}}}{\partial x_3}; \\
\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} &= -\frac{\sqrt{E_{11}}}{E_{22}} \frac{\partial \sqrt{E_{11}}}{\partial x_2}; & \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} &= \frac{1}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{22}}}{\partial x_1}; & \left\{ \begin{matrix} 13 \\ 2 \end{matrix} \right\} &= 0; \\
\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} &= \frac{1}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{22}}}{\partial x_2}; & \left\{ \begin{matrix} 23 \\ 2 \end{matrix} \right\} &= \frac{1}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{22}}}{\partial x_3}; & \left\{ \begin{matrix} 33 \\ 2 \end{matrix} \right\} &= -\frac{\sqrt{E_{33}}}{E_{22}} \frac{\partial \sqrt{E_{33}}}{\partial x_2}; \\
\left\{ \begin{matrix} 11 \\ 3 \end{matrix} \right\} &= -\frac{\sqrt{E_{11}}}{E_{33}} \frac{\partial \sqrt{E_{11}}}{\partial x_3}; & \left\{ \begin{matrix} 12 \\ 3 \end{matrix} \right\} &= 0; & \left\{ \begin{matrix} 13 \\ 3 \end{matrix} \right\} &= \frac{1}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{33}}}{\partial x_2}; \\
\left\{ \begin{matrix} 22 \\ 3 \end{matrix} \right\} &= -\frac{\sqrt{E_{22}}}{E_{33}} \frac{\partial \sqrt{E_{22}}}{\partial x_3}; & \left\{ \begin{matrix} 23 \\ 3 \end{matrix} \right\} &= \frac{1}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{33}}}{\partial x_2}; & \left\{ \begin{matrix} 33 \\ 3 \end{matrix} \right\} &= \frac{1}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{33}}}{\partial x_2}.
\end{aligned}$$

§5. The *tangent hyperplane* is determined by any three lines not in the same plane lying in it and passing through a point. In particular, it is determined by the three coordinate lines. Defining then the normal to the hypersurface at any point as the line perpendicular to the tangent hyperplane, we have to express the fact that the line whose direction cosines are

$$Y_1, Y_2, Y_3, Y_4,$$

is orthogonal to the coordinate lines. We have then to determine these direction cosines the three equations

$$\sum_i Y_i \frac{\partial y_i}{\partial x_r} = 0, \quad r = 1, 2, 3.$$

These give at once

$$Y_i = \frac{1}{\sqrt{\Delta}} \frac{\partial (y_m y_p y_q)}{\partial (x_1 x_2 x_3)}, \quad l \neq m \neq p \neq q, \quad (1)$$

where Δ is the discriminant of the linear element expressed in the $E_{r,s}$. Introducing the second ternary differential quadratic form

$$\begin{aligned} \phi &= - \sum dY dy \\ &= \sum_{r,s} D_{rs} dx_r dx_s, \end{aligned} \quad (2)$$

we at once find

$$D_{rs} = \sum Y \frac{\partial^2 y}{\partial x_r \partial x_s} = - \sum \frac{\partial Y}{\partial x_r} \frac{\partial y}{\partial x_s} = - \sum \frac{\partial Y}{\partial x_s} \frac{\partial y}{\partial x_r}. \quad (3)$$

by means of (1) these can be written

$$D_{rs} = \frac{1}{\sqrt{\Delta}} \begin{vmatrix} \frac{\partial^2 y_1}{\partial x_r \partial x_s} & \frac{\partial^2 y_2}{\partial x_r \partial x_s} & \frac{\partial^2 y_3}{\partial x_r \partial x_s} & \frac{\partial^2 y_4}{\partial x_r \partial x_s} \\ \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \frac{\partial y_3}{\partial x_1} & \frac{\partial y_4}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_4}{\partial x_2} \\ \frac{\partial y_1}{\partial x_3} & \frac{\partial y_2}{\partial x_3} & \frac{\partial y_3}{\partial x_3} & \frac{\partial y_4}{\partial x_3} \end{vmatrix} \quad (3')$$

If, now, A_1, A_2, A_3, A_4 be any four functions of x_1, x_2, x_3 , then $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ can be so determined that the four equations

$$A_i = \sum \alpha_r \frac{\partial y_i}{\partial x_r} + \alpha_4 Y_i, \quad i = 1, 2, 3, 4 \quad (a)$$

are satisfied since the determinant of the system equals $\sqrt{\Delta}$ and is therefore not zero. By exactly the same reasoning as that employed in the case of two parameters,* we arrive at the following system of equations satisfied by the coordinates

$$\frac{\partial^2 y_i}{\partial x_r \partial x_s} = \sum_t \left\{ \begin{matrix} rs \\ t \end{matrix} \right\} \frac{\partial y_i}{\partial x_t} + D_{rs} Y_i, \quad (I)$$

and the following system satisfied by the direction cosines of the normal:

$$\frac{\partial Y_i}{\partial x_r} = -\frac{1}{\Delta} \sum_s \left| D_{1r}^{(s)} \right| \frac{\partial y_i}{\partial x_s}, \quad (II)$$

where

$$\left| D_{1r}^{(s)} \right| = \begin{vmatrix} E_{1, s-1} & D_{1r} & E_{1, s+1} \\ E_{2, s-1} & D_{2r} & E_{2, s+1} \\ E_{3, s-1} & D_{3r} & E_{3, s+1} \end{vmatrix}$$

The relations connecting the coefficients of the two forms are now found by writing the conditions of integrability of (I) and (II). These are for (I)

$$\frac{\partial}{\partial x_t} \left(\frac{\partial^2 y_i}{\partial x_r \partial x_s} \right) = \frac{\partial}{\partial x_r} \left(\frac{\partial^2 y_i}{\partial x_s \partial x_t} \right),$$

which give

$$\begin{aligned} & \sum_p^3 \left[\left\{ \begin{matrix} rs \\ p \end{matrix} \right\} \frac{\partial^2 y_i}{\partial x_p \partial x_t} + \frac{\partial}{\partial x_t} \left\{ \begin{matrix} rs \\ p \end{matrix} \right\} \frac{\partial y_i}{\partial x_p} + Y_i \frac{\partial D_{rs}}{\partial x_r} + D_{rs} \frac{\partial Y_i}{\partial x_t} \right] \\ &= \sum_p^3 \left[\left\{ \begin{matrix} st \\ p \end{matrix} \right\} \frac{\partial^2 y_i}{\partial x_p \partial x_r} + \frac{\partial}{\partial x_r} \left\{ \begin{matrix} st \\ p \end{matrix} \right\} \frac{\partial y_i}{\partial x_p} + Y_i \frac{\partial D_{st}}{\partial x_r} + D_{st} \frac{\partial Y_i}{\partial x_r} \right]. \end{aligned}$$

Substituting for $\frac{\partial^2 y_i}{\partial x_r \partial x_s}$ and $\frac{\partial Y_i}{\partial x_t}$ from (I) and (II) we have

$$\begin{aligned} & \sum_{pq}^3 \left[\left(\left\{ \begin{matrix} rs \\ p \end{matrix} \right\} \left\{ \begin{matrix} pt \\ q \end{matrix} \right\} - \left\{ \begin{matrix} st \\ p \end{matrix} \right\} \left\{ \begin{matrix} pr \\ q \end{matrix} \right\} + \frac{\partial}{\partial x_t} \left\{ \begin{matrix} rs \\ q \end{matrix} \right\} - \frac{\partial}{\partial x_r} \left\{ \begin{matrix} st \\ q \end{matrix} \right\} - \frac{D_{rs}}{\Delta} \left| D_{1t}^{(q)} \right| + \frac{D_{st}}{\Delta} \left| D_{1r}^{(q)} \right| \right) \frac{\partial y_i}{\partial x_q} \right. \\ & \quad \left. + \left(\frac{\partial D_{rs}}{\partial x_t} - \frac{\partial D_{st}}{\partial x_r} + D_{qt} \left\{ \begin{matrix} rs \\ q \end{matrix} \right\} - D_{qr} \left\{ \begin{matrix} st \\ q \end{matrix} \right\} \right) Y_i \right] = 0. \quad (b) \end{aligned}$$

This system is again of type (a) with the first member zero, and hence the coefficients must separately vanish. Introducing the symbol of four indices

$$\left\{ \begin{matrix} ps, rt \\ p \end{matrix} \right\} = \frac{\partial}{\partial x_t} \left\{ \begin{matrix} rs \\ p \end{matrix} \right\} - \frac{\partial}{\partial x_r} \left\{ \begin{matrix} st \\ p \end{matrix} \right\} + \sum_q \left(\left\{ \begin{matrix} rs \\ p \end{matrix} \right\} \left\{ \begin{matrix} pt \\ q \end{matrix} \right\} - \left\{ \begin{matrix} st \\ p \end{matrix} \right\} \left\{ \begin{matrix} pr \\ q \end{matrix} \right\} \right),$$

* Bianchi, *Lezioni*, p. 87.

we have the systems

$$\left\{sq, rt\right\} - \frac{D_{rs}}{\Delta} \left| D_{1t}^{(q)} \right| + \frac{D_{st}}{\Delta} \left| D_{1r}^{(q)} \right| = 0, \quad (\text{III})$$

$$\frac{\partial D_{rs}}{\partial x_t} - \frac{\partial D_{st}}{\partial x_r} + \sum_p \left(D_{pt} \left\{ \begin{smallmatrix} rs \\ p \end{smallmatrix} \right\} - D_{pr} \left\{ \begin{smallmatrix} st \\ p \end{smallmatrix} \right\} \right) = 0, \quad (\text{IV})$$

$$(p, q, r, s, t = 1, 2, 3).$$

These equations can be written in a different form.

In the identities

$$(s p r t) = \sum_q a_{qp} \{s q r t\},$$

substituting for $\{s q r t\}$ from (III), we have at once, after reduction,

$$(s p r t) = D_{rs} D_{pt} - D_{st} D_{pr}. \quad (\text{III}')$$

Since six only of the symbols $(s p r t)$ are independent and different from zero, the coefficients of the second fundamental form are completely determined, the eight relations (IV), which may be looked upon as the generalization* of the Mainardi-Codazzi equations.

From (IV) we have at once

$$\frac{\partial D_{lp}}{\partial x_m} - \frac{\partial D_{lm}}{\partial x_p} + D_{lm} \left\{ \begin{smallmatrix} lp \\ l \end{smallmatrix} \right\} - D_{lp} \left\{ \begin{smallmatrix} lm \\ l \end{smallmatrix} \right\} + \sum_r' \left(D_{rm} \left\{ \begin{smallmatrix} lp \\ r \end{smallmatrix} \right\} - D_{rp} \left\{ \begin{smallmatrix} lm \\ r \end{smallmatrix} \right\} \right) = 0, \quad (\text{c})$$

where, in \sum' , $r = l$ is excluded.

Observing that

$$\frac{\partial \log \sqrt{a}}{\partial x_i} = \sum_{i\kappa} A_{i\kappa} \left[\begin{smallmatrix} i\ell \\ \kappa \end{smallmatrix} \right]$$

we have

$$\frac{\partial \left(\frac{1}{\sqrt{a}} \right)}{\partial x_i} = \sum_i \frac{1}{\sqrt{a}} \left\{ \begin{smallmatrix} i\ell \\ i \end{smallmatrix} \right\}.$$

* Ricci, Acc. dei L. Rend. 2, 1895, §V, p. 320; Cesáro, "Geometria Intrinseca," p. 238.

Hence

$$D_{lp} \frac{\partial \left(\frac{1}{\sqrt{\Delta}} \right)}{\partial x_m} - D_{lm} \frac{\partial \left(\frac{1}{\sqrt{\Delta}} \right)}{\partial x_p} + \frac{1}{\sqrt{\Delta}} \left[\left(D_{lp} \left\{ \begin{smallmatrix} lm \\ l \end{smallmatrix} \right\} - D_{lm} \left\{ \begin{smallmatrix} lp \\ l \end{smallmatrix} \right\} \right) + \sum' \left(D_{lp} \left\{ \begin{smallmatrix} rm \\ r \end{smallmatrix} \right\} - D_{lm} \left\{ \begin{smallmatrix} rp \\ r \end{smallmatrix} \right\} \right) \right] = 0. \quad (d)$$

Adding (c) and (d),

$$\frac{\partial \left(\frac{D_{lp}}{\sqrt{\Delta}} \right)}{\partial x_m} - \frac{\partial \left(\frac{D_{lm}}{\sqrt{\Delta}} \right)}{\partial x_p} + \frac{1}{\sqrt{\Delta}} \left[\sum' \left(D_{rm} \left\{ \begin{smallmatrix} lp \\ r \end{smallmatrix} \right\} - D_{rp} \left\{ \begin{smallmatrix} lm \\ r \end{smallmatrix} \right\} + D_{lp} \left\{ \begin{smallmatrix} rm \\ r \end{smallmatrix} \right\} - D_{lm} \left\{ \begin{smallmatrix} rp \\ r \end{smallmatrix} \right\} \right) \right] = 0. \quad (IV')$$

Since the D_{rs} are immediately expressible in terms of the E_{rs} and their derivatives by (III'), we have here eight differential relations satisfied by the coefficients of the linear element of the hypersurface. Hence we may say, *in order that the DEFINITE ternary differential quadratic form*

$$f = \sum_{rs} a_{rs} dx_r dx_s$$

shall represent the linear element of a hypersurface, the coefficients must satisfy (IV').

§6. *Conversely, given the ternary differential quadratic form*

$$f = \sum_{rs} a_{rs} dx_r dx_s,$$

which is DEFINITE and whose coefficients satisfy (IV'), there exists a UNIQUE hypersurface admitting f as linear element, and in order to effectively obtain this hypersurface, it is necessary to integrate three generalized Riccati equations.

Suppose the hypersurface, whose existence and uniqueness we wish to determine under the hypotheses above, to be referred to its *lines of curvature*, and consider at every point the *tetraprectangular tetrahedroid* formed by the tangents to the positive directions of these three lines and the normal. Let $Y_{\lambda\mu}$ ($\lambda = 1, 2, 3, 4$) be the direction cosines of the tangent to the line x_μ and $Y_{\lambda 4}$, those of the normal. Then

$$Y_{\lambda\mu} = \frac{1}{\sqrt{E_{\mu\mu}}} \frac{\partial y_\lambda}{\partial x_\mu},$$

$$Y_{\lambda 4} = Y_\lambda.$$

From (I) and (II), p. 254, substituting for the Christoffel symbols their values from p. 252, we have

$$\begin{aligned}\frac{\partial Y_{11}}{\partial x_1} &= -\frac{Y_{12}}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{11}}}{\partial x_2} - \frac{Y_{13}}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{11}}}{\partial x_3} + \frac{D_{11}}{\sqrt{E_{11}}} Y_{14}; \\ \frac{\partial Y_{11}}{\partial x_2} &= \frac{Y_{12}}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{22}}}{\partial x_1}; \quad \frac{\partial Y_{11}}{\partial x_3} = \frac{Y_{13}}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{33}}}{\partial x_1}; \\ \frac{\partial Y_{12}}{\partial x_1} &= \frac{Y_{11}}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{11}}}{\partial x_2}; \quad \frac{\partial Y_{12}}{\partial x_2} = \frac{Y_{13}}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{33}}}{\partial x_2}; \\ \frac{\partial Y_{12}}{\partial x_3} &= -\frac{Y_{11}}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{22}}}{\partial x_1} - \frac{Y_{13}}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{22}}}{\partial x_3} + \frac{D_{22}}{\sqrt{E_{22}}} Y_{14}; \\ \frac{\partial Y_{13}}{\partial x_1} &= \frac{Y_{11}}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{11}}}{\partial x_3}; \quad \frac{\partial Y_{13}}{\partial x_2} = \frac{Y_{12}}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{22}}}{\partial x_3}; \\ \frac{\partial Y_{13}}{\partial x_3} &= -\frac{Y_{11}}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{33}}}{\partial x_1} - \frac{Y_{12}}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{33}}}{\partial x_2} + \frac{D_{33}}{\sqrt{E_{33}}} Y_{14}; \\ \frac{\partial Y_{14}}{\partial x_1} &= -\frac{D_{11}}{\sqrt{E_{11}}} Y_{11}; \quad \frac{\partial Y_{14}}{\partial x_2} = -\frac{D_{22}}{\sqrt{E_{22}}} Y_{12}; \quad \frac{\partial Y_{14}}{\partial x_3} = -\frac{D_{33}}{\sqrt{E_{33}}} Y_{13}.\end{aligned}$$

The four functions $Y_{\lambda\mu}$ ($\mu = 1, 2, 3, 4$) then satisfy the four simultaneous linear homogeneous total differential equations

$$\left. \begin{aligned}dY_{\lambda 1} &= \left\{ -\frac{Y_{\lambda 2}}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{11}}}{\partial x_2} - \frac{Y_{\lambda 3}}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{11}}}{\partial x_3} + \frac{D_{11}}{\sqrt{E_{11}}} Y_{\lambda 4} \right\} dx_1 \\ &\quad + \frac{Y_{\lambda 2}}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{22}}}{\partial x_1} dx_2 + \frac{Y_{\lambda 3}}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{33}}}{\partial x_1} dx_3, \\ dY_{\lambda 2} &= \frac{Y_{\lambda 1}}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{11}}}{\partial x_2} dx_1 \\ &\quad + \left\{ -\frac{Y_{\lambda 1}}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{22}}}{\partial x_1} - \frac{Y_{\lambda 3}}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{22}}}{\partial x_3} + \frac{D_{22}}{\sqrt{E_{22}}} Y_{\lambda 4} \right\} dx_2 \\ &\quad + \frac{Y_{\lambda 3}}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{33}}}{\partial x_2} dx_3, \\ dY_{\lambda 3} &= \frac{Y_{\lambda 1}}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{11}}}{\partial x_3} dx_1 + \frac{Y_{\lambda 2}}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{22}}}{\partial x_3} dx_2 \\ &\quad + \left\{ -\frac{Y_{\lambda 1}}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{33}}}{\partial x_1} - \frac{Y_{\lambda 2}}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{33}}}{\partial x_2} + \frac{D_{33}}{\sqrt{E_{33}}} Y_{\lambda 4} \right\} dx_3, \\ dY_{\lambda 4} &= -\frac{D_{11}}{\sqrt{E_{11}}} Y_{\lambda 1} dx_1 - \frac{D_{22}}{\sqrt{E_{22}}} Y_{\lambda 2} dx_2 - \frac{D_{33}}{\sqrt{E_{33}}} Y_{\lambda 3} dx_3.\end{aligned} \right\} (4)$$

This is an illimitably integrable system in virtue of equations (III) and (IV),

remembering that in the case of the hypersurface referred to its lines of curvature,

$$E_{rs} = D_{rs} = 0, \quad r \neq s.$$

Hence there exists an integral system, and a single one, which, for the initial values of the variables $x_i = x_i^0$, reduces to arbitrarily given initial values.

If $Y_{\lambda\mu}$ and $Y'_{\lambda\mu}$ are two integral systems, then

$$\sum_1^4 Y_{\lambda\mu} Y'_{\lambda\mu} = \text{const.}$$

Let $Y_{\lambda\mu}$ ($\lambda, \mu = 1, 2, 3, 4$) be four integral systems of (4) which for initial values $x_i = x_i^0$, reduce to the sixteen coefficients of an orthogonal substitution. Then it follows from the observation above, that for all values of the variables will these sixteen quantities be the coefficients of an orthogonal substitution, and in particular,

$$\begin{aligned} \sum_1^4 Y_{\lambda\mu}^2 &= 1, \\ \sum_1^4 Y_{\lambda\mu} Y_{\lambda\nu} &= 0, \end{aligned} \quad (\mu \neq \nu).$$

From (4) it is easily seen that the four expressions

$$\sum_1^3 \sqrt{E_{ii}} Y_{\lambda i} dx_i, \quad (\lambda = 1, 2, 3, 4)$$

are exact differentials, and writing

$$y_\lambda = \int \sum_1^3 \sqrt{E_{ii}} Y_{\lambda i} dx_i,$$

we have a hypersurface with the given fundamental forms.

§7. The system (4) is identical with systems (34), (34'), (34'') found by Professor Craig* when $\alpha, \beta, \gamma, \delta$ are replaced by $Y_{\lambda 1}, Y_{\lambda 2}, Y_{\lambda 3}, Y_{\lambda 4}$ and the p_{ij} by the coefficients in (4). Now Professor Craig has shown that the integration of (34) can be reduced to the integration of a generalized† Riccati equation

* Amer. Jour., Vol. XX, No. 2, p. 145.

† L. c., p. 141.

and in the same way (4) can be reduced to the integration of the three generalized Riccati equations

$$\left. \begin{aligned} \frac{\partial \lambda}{\partial x_1} &= -\frac{1}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{11}}}{\partial x_2} \mu - \frac{1}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{11}}}{\partial x_3} \nu + \frac{\kappa^2 - 1}{2} \cdot \frac{D_{11}}{\sqrt{E_{11}}} - \lambda^2 \frac{D_{11}}{\sqrt{E_{11}}}, \\ \frac{\partial \mu}{\partial x_1} &= \frac{1}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{11}}}{\partial x_2} \lambda \\ \frac{\partial \nu}{\partial x_1} &= \frac{1}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{11}}}{\partial x_3} \lambda \end{aligned} \right\} - \lambda \mu \frac{D_{11}}{\sqrt{E_{11}}}, - \lambda \nu \frac{D_{11}}{\sqrt{E_{11}}},$$

$$\left. \begin{aligned} \frac{\partial \lambda}{\partial x_2} &= \frac{1}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{22}}}{\partial x_2} \mu \\ \frac{\partial \mu}{\partial x_2} &= -\frac{1}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{22}}}{\partial x_1} \lambda - \frac{1}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{22}}}{\partial x_3} \nu + \frac{\kappa^2 - 1}{2} \frac{D_{22}}{\sqrt{E_{22}}} - \mu^2 \frac{D_{22}}{\sqrt{E_{22}}}, \\ \frac{\partial \nu}{\partial x_2} &= \frac{1}{\sqrt{E_{33}}} \frac{\partial \sqrt{E_{22}}}{\partial x_3} \mu \end{aligned} \right\} - \mu \lambda \frac{D_{22}}{\sqrt{E_{22}}}, - \mu \nu \frac{D_{22}}{\sqrt{E_{22}}},$$

$$\left. \begin{aligned} \frac{\partial \lambda}{\partial x_3} &= \frac{1}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{22}}}{\partial x_1} \nu \\ \frac{\partial \mu}{\partial x_3} &= \frac{1}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{33}}}{\partial x_2} \nu \\ \frac{\partial \nu}{\partial x_3} &= -\frac{1}{\sqrt{E_{11}}} \frac{\partial \sqrt{E_{33}}}{\partial x_1} \lambda - \frac{1}{\sqrt{E_{22}}} \frac{\partial \sqrt{E_{33}}}{\partial x_2} \mu + \frac{\kappa^2 - 1}{2} \frac{D_{33}}{\sqrt{E_{33}}} - \nu^2 \frac{D_{33}}{\sqrt{E_{33}}} \end{aligned} \right\} - \nu \lambda \frac{D_{33}}{\sqrt{E_{33}}}, - \nu \mu \frac{D_{33}}{\sqrt{E_{33}}}.$$

By the substitutions

$$Y_{\lambda 1} = \frac{2\lambda}{\kappa^2 + 1}; \quad Y_{\lambda 2} = \frac{2\mu}{\kappa^2 + 1}; \quad Y_{\lambda 3} = \frac{2\nu}{\kappa^2 + 1}; \quad Y_{\lambda 4} = \frac{\kappa^2 - 1}{\kappa^2 + 1};$$

$$\kappa^2 = \lambda^2 + \mu^2 + \nu^2.$$

§7. Here we have chosen the special tetrahedroid formed by the tangents to the lines of curvature and the normal, but any other *tetrarectangular tetrahedroid* might have been taken and we should have arrived at a set of equations similar to (4), illimitably integrable in virtue of equations (III) and (IV), and these would have led to three generalized Riccati equations of the form

$$\left. \begin{aligned} \frac{\partial \lambda}{\partial x_i} &= a_{1i} \mu + b_{1i} \nu + \frac{\kappa^2 - 1}{2} c_{1i} + \lambda (a'_{1i} \lambda + b'_{1i} \mu + c'_{1i} \nu), \\ \frac{\partial \mu}{\partial x_i} &= a_{2i} \lambda + b_{2i} \nu + \frac{\kappa^2 - 1}{2} c_{2i} + \mu (a'_{2i} \lambda + b'_{2i} \nu + c'_{2i} \nu), \\ \frac{\partial \nu}{\partial x_i} &= a_{3i} \lambda + b_{3i} \mu + \frac{\kappa^2 - 1}{2} c_{3i} + \nu (a'_{3i} \lambda + b'_{3i} \mu + c'_{3i} \nu). \end{aligned} \right\}$$

§8. It is interesting to note that the equations (III) and (IV), which must be satisfied in order that

$$f = \sum_{r,s}^3 E_{rs} dx_r dx_s$$

shall represent the linear element of a hypersurface, are exactly the conditions that f shall be irreducible and of the first class.* From the definition of D_{rs} we observe that

$$D_{rs} = (rs),$$

and equations (III'), p. 255, become exactly equations (I), p. 153, and substituting $D_{rs} = (rs)$ in equations (IV), p. 255, we have (II), p. 153.

§9. From equations (III'), p. 255, it follows that the second fundamental form is completely determined when the first is given, and from the theorem on p. 256, it follows that in this case the hypersurface is *uniquely* determined to within motion in hyperspace. Hence the property possessed by surfaces in Euclidian space of three dimensions of being deformed without alteration of the linear element, cannot be extended to hypersurfaces. We thus come upon a theorem noted by Beez,† and further put in evidence by Ricci.‡

* Ricci, "Principii di una teoria," etc., p. 151.

† Cesáro, "Lezioni di Geometria Intrinseca," p. 247.

‡ "Principii di una teoria," etc., p. 163.